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1999 J. Phys. A: Math. Gen. 32 L159

(http://iopscience.iop.org/0305-4470/32/13/003)

J. Phys. A: Math. Gen. **32** (1999) L159–L163. Printed in the UK PII: S0305-4470(99)00492-8

LETTER TO THE EDITOR

Fluctuation formula for complex random matrices

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Received 22 December 1998

Abstract. A Gaussian fluctuation formula is proved for linear statistics of complex random matrices in the case where the statistic is rotationally invariant. For a general linear statistic without this symmetry, Coulomb gas theory is used to predict that the distribution will again be a Gaussian, with a specific mean and variance. The variance splits naturally into a bulk and surface contibution, the latter resulting from the long-range correlations at the boundary of the support of the eigenvalue density.

The phenomenom of universal conductance fluctuations in mesoscopic wires (see e.g. [1]) has provided the motivation for a number of theoretical studies into fluctuation formulae for linear statistics in random matrix ensembles [2]. To understand the reason for this, we first recall that the striking feature of the conductance fluctuations is that they remain of order unity even though the conductance itself is proportional to the number of channels N. Now, in random matrix models of this effect, the conductance can be written as a linear statistic of a certain random matrix ensemble (we recall that A is said to be a linear statistic of the eigenvalues λ_j if it can be written in the form $A = \sum_{j=1}^{N} a(\lambda_j)$ for some function a). In this setting, the theoretical explanation for the phenomenum of universal conductance fluctuations is as an example of a universal fluctuation formula in random matrix theory, the first example of which was given in the pioneering work of Dyson and Mehta [3].

For random matrix ensembles in which the support of the density is one dimensional, for example Hermitian or unitary random matrices, (Gaussian) fluctuation formulae are now well understood both at a heuristic (see references cited above) and rigorous level [4–6]. It is the purpose of this letter to initiate the study of fluctuation formulae in complex random matrices [7–9], for which the eigenvalues uniformly fill a disc or ellipse in the complex plane. We remark that complex random matrices have occured in recent physical studies of the localization–delocalization transition in non-Hermitian quantum mechanics [10] and chiral symmetry breaking in lattice QCD [11]. The distribution of a linear statistic is then an observable quantity after averaging over many random copies of these systems.

To begin, we recall [7] that for a random matrix with complex elements $u_{jk} + \mathrm{i}v_{jk}$ independently distributed with Gaussian distribution $\frac{c}{\pi}\mathrm{e}^{-c(|u_{jk}|^2+|v_{jk}|^2)}$, the corresponding probability distribution of the eigenvalues $\lambda_j = x_j + \mathrm{i}y_j$ is proportional to

$$\prod_{j=1}^{N} \mathrm{e}^{-c|\vec{r}_j|^2} \prod_{1\leqslant j<k\leqslant N} |\vec{r}_j - \vec{r}_k|^2 \tag{1}$$

where $\vec{r}_j = (x_j, y_j)$. Furthermore, to leading order, the support of the density of the eigenvalues is the disc of radius $\sqrt{N/c}$. For the purpose of studying fluctuation formulae it is convenient to choose $c = N$ so that the support of the density is the unit disc. The Fourier transform of the distribution of $\Pr(A = u)$ is then given by

$$\tilde{P}(k) = \frac{\prod_{l=1}^{N} \int_{\mathbf{R}^2} \mathrm{d}\vec{r}_l \, \mathrm{e}^{-N|\vec{r}_l|^2 + \mathrm{i}ka(\vec{r}_l)} \prod_{1 \leqslant j < k \leqslant N} |\vec{r}_k - \vec{r}_j|^2}{\prod_{l=1}^{N} \int_{\mathbf{R}^2} \mathrm{d}\vec{r}_l \, \mathrm{e}^{-N|\vec{r}_l|^2} \prod_{1 \leqslant j < k \leqslant N} |\vec{r}_k - \vec{r}_j|^2}. \tag{2}$$

Suppose now that the linear statistic is of the form $A = \sum_{j=1}^{N} a(|\vec{r}_j|)$, so that the function a only depends on the distance from the origin. Introducing polar coordinates and using the Vandermonde determinant expansion of $\prod_{1 \leqslant j < k \leqslant N}(z_k - z_j)$ the integrals in (2) can be evaluated with the result

$$\tilde{P}(k) = \frac{\prod_{l=1}^{N} \int_0^\infty \mathrm{e}^{-s} s^{l-1} \mathrm{e}^{\mathrm{i}ka(\sqrt{s/N})} \, \mathrm{d}s}{\prod_{l=1}^{N} \int_0^\infty \mathrm{e}^{-s} s^{l-1} \, \mathrm{d}s}. \tag{3}$$

This is the exact expression for finite N. To obtain its form for $N \to \infty$, we change variables $s \to ls$ and expand the integrand about its large-l maximum at $s = 1$. A straightforward calculation then gives

$$\tilde{P}(k) \sim \prod_{l=1}^{N} \mathrm{e}^{\mathrm{i}ka(\sqrt{l/N})} \mathrm{e}^{-k^2(a'(\sqrt{l/N}))^2} \sim \mathrm{e}^{\mathrm{i}k\mu} \mathrm{e}^{-k^2\sigma^2/2} \tag{4}$$

with

$$\mu = 2N \int_0^1 r a(r) \, \mathrm{d}r \qquad \sigma^2 = \tfrac{1}{2} \int_0^1 r(a'(r))^2 \, \mathrm{d}r. \tag{5}$$

Thus the distribution of A is a Gaussian with mean and variance as given by (5). Note in particular that the variance is O(1).

Next we address the more general situation in which a is not rotationally invariant. To make progress we must proceed heuristically. The p.d.f. (1) can be interpreted as the Boltzmann factor of the two-dimensional one-component plasma (2dOCP) at the special value of the coupling $\Gamma = 2$, [12]. Using linear response theory and macroscopic electrostatics, it is possible to argue [13, 14] that in general the distribution of a linear statistic in a classical Coulomb system in the conductive phase will be Gaussian (this assumes also that the random function varies over macroscopic distances relative to the interparticle spacing). The Gaussian distribution is uniquely characterized by its mean and variance. But independent of the underlying distribution, these quantities are given by

$$\mu = \frac{N}{\pi} \int_\Lambda \mathrm{d}\vec{r} \, a(\vec{r}), \qquad \sigma^2 = \int_\Lambda \mathrm{d}\vec{r}_1 \, a(\vec{r}_1) \int_\Lambda \mathrm{d}\vec{r}_2 \, a(\vec{r}_2) S(\vec{r}_1, \vec{r}_2) \tag{6}$$

where $S(\vec{r}_1, \vec{r}_2)$ is given in terms of the truncated two-particle distribution function by $S(\vec{r}_1, \vec{r}_2) = \rho_{(2)}^T(\vec{r}_1, \vec{r}_2) + \frac{N}{\pi}\delta(\vec{r} - \vec{r}\,')$, $\frac{N}{\pi}$ is the particle density and Λ denotes the unit disc. We see immediately from the formula for μ in (6) that the formula for μ in (5) is reclaimed if $a(\vec{r}) = a(|\vec{r}|)$. More challenging is to reproduce the formula for σ^2, and to proceed to generalize this formula for general $a(\vec{r})$.

This task can be undertaken by again appealing to Coulomb gas theory. In the infinite density limit the function $S(\vec{r}_1, \vec{r}_2)$ in (6) for the 2dOCP with general coupling Γ is expected to have the *bulk* universal form [14]

$$\begin{aligned} S_{\mathrm{bulk}}(\vec{r}_1, \vec{r}_2) &= -\frac{1}{2\pi\Gamma} \nabla^2 \delta(\vec{r}_1 - \vec{r}_2) \\ &= \frac{1}{2\pi\Gamma} \left(\frac{\partial}{\partial x^{(1)}} + \mathrm{i} \frac{\partial}{\partial y^{(1)}} \right) \left(\frac{\partial}{\partial x^{(2)}} - \mathrm{i} \frac{\partial}{\partial y^{(2)}} \right) \delta(\vec{r}_1 - \vec{r}_2). \end{aligned} \tag{7}$$

At $\Gamma = 2$ this can be checked from the $\rho \to \infty$ limit of the exact formula [7] $S_{\rm bulk}(\vec{r}_1, \vec{r}_2) = -\rho^2 {\rm e}^{-\pi\rho|\vec{r}_1-\vec{r}_2|^2} + \rho\delta(\vec{r}_1 - \vec{r}_2)$. Substituting (7) in (6), and integrating by parts (ignoring possible boundary terms, which are separately treated below) gives

$$\sigma^2_{\rm bulk} = \frac{1}{2\pi\Gamma}\int_\Lambda {\rm d}x\,{\rm d}y\,\left(\left(\frac{\partial a(x,y)}{\partial x}\right)^2 + \left(\frac{\partial a(x,y)}{\partial y}\right)^2\right). \tag{8}$$

In the special case $a(\vec{r}) = a(|\vec{r}|)$, $\Gamma = 2$, (8) reproduces the result (5) for σ^2.

The crucial difference between the case of general $a(\vec{r})$ and the rotationally invariant case $a(\vec{r}) = a(|\vec{r}|)$ is that in the latter case σ^2 contains a contribution from the surface correlations of the same order (O(1)) as the contribution from the bulk correlations. This effect, due to the long-range nature of the correlations at the boundary of Coulomb systems [15], was first noted by Choquard [16] and collaborators, who studied the variance of the dipole moment ($a(\vec{r}) = x$) for classical Coulomb systems. Indeed the variance of this statistic was used to compute from microscopic statistical mechanics the macroscopic *shape dependent* dielectric susceptibility of the Coulomb system.

Like in the bulk, the correlation $S(\vec{r}_1, \vec{r}_2)$ has a universal form for $\vec{r}_1$ and $\vec{r}_2$ at the surface. However, unlike the situation in the bulk, this correlation is long ranged and shape dependent. For Λ a unit disc, the universal form is [14]

$$S_{\rm surface}((r_1,\theta_1),(r_2,\theta_2)) = -\frac{1}{2\pi^2\Gamma}\left(\frac{\partial^2}{\partial\theta_1\partial\theta_2}\log\left|\sin\frac{\theta_1-\theta_2}{2}\right|\right)\delta(r_1-1)\delta(r_2-1) \tag{9}$$

where polar coordinates have been introduced. At $\Gamma = 2$ this form can be derived explicitly from the exact evaluation of $S(\vec{r}_1, \vec{r}_2)$ in the finite system [16]. Substituting in (6) gives

$$\begin{aligned}\sigma^2_{\rm surface} &= -\frac{1}{2\pi^2\Gamma}\int_0^{2\pi}{\rm d}\theta_1\,\left(\frac{\partial}{\partial\theta_1}a(1,\theta_1)\right)\int_0^{2\pi}{\rm d}\theta_2\,\left(\frac{\partial}{\partial\theta_2}a(1,\theta_2)\right)\log\left|\sin\frac{\theta_1-\theta_2}{2}\right| \\ &= \frac{2}{\Gamma}\sum_{n=1}^{\infty} n|a_n|^2 \qquad a(1,\theta) = \sum_{n=-\infty}^{\infty} a_n {\rm e}^{{\rm i}n\theta}.\end{aligned} \tag{10}$$

This quantity vanishes for $a(\vec{r}) = a(|\vec{r}|)$.

Consider now a more general ensemble of complex random matrices [8], in which the members, J say, are of the form $J = H + {\rm i}vA$. Here H and A are Gaussian Hermitian random matrices with joint p.d.fs for the elements proportional to $\exp(-\frac{N}{1+\tau}\,{\rm tr}\,X^2)$ ($X = H, A$ and $\tau = (1-v^2)/(1+v^2)$). The corresponding eigenvalue p.d.f. is proportional to

$$\exp\left(-N\sum_{j=1}^{N}\left(\frac{x_j^2}{1+\tau} + \frac{y_j^2}{1-\tau}\right)\right)\prod_{1\leqslant j<k\leqslant N}|\vec{r}_j - \vec{r}_k|^2 \tag{11}$$

(note that in the case $\tau = 0$ (11) agrees with (1)), and to leading order the support of the eigenvalue density consists of an ellipse with semi-axes $A = (1+\tau)$, $B = (1-\tau)$. The eigenvalue density itself is uniform and thus has the value $N/(\pi(1-\tau^2)$ inside the ellipse.

The p.d.f. (11) can be interpreted as the Boltzmann factor of the 2dOCP at $\Gamma = 2$ in a quadrupolar field [17, 18]. Thus, Coulomb gas theory gives that as $N \to \infty$ (infinite density limit) the distribution of a linear statistic will again be Gaussian. The mean will be given as in (6), but with Λ now the ellipse specifying the support of the eigenvalues, and the factor N/π which represents the eigenvalue density replaced by $N/(\pi(1-\tau^2))$. With Λ the ellipse, the bulk contribution to the variance is again given by (8). For the surface contribution, we require the fact that the universal form of the surface correlation at the boundary of an ellipse is [16, 18]

$$S_{\text{surface}}((\xi_1,\eta_1),(\xi_2,\eta_2)) = -\frac{1}{2\pi^2\Gamma h(\xi_b,\eta_1)h(\xi_b,\eta_2)}\left(\frac{\partial^2}{\partial\eta_1\partial\eta_2}\log\left|\sin\frac{\eta_1-\eta_2}{2}\right|\right)\delta(\xi_1-\xi_b)\delta(\xi_2-\xi_b) \tag{12}$$

where (ξ,η) are elliptic coordinates, specified by $x+\mathrm{i}y = \cosh(\xi+\mathrm{i}\eta)$, and $h(\xi_b,\eta)\,\mathrm{d}\eta$ gives the differential surface element. As in the disc case (9), this form has been explicitly verified [18] from the known exact expression for $S(\vec{r}_1,\vec{r}_2)$ in the finite system. Since the semi-axes are specified by $A=\cosh\xi_b$, $B=\sinh\xi_b$, ξ_b is related to τ by $\tanh\xi_b=(1-\tau)/(1+\tau)$. Substituting (12) in (6) gives

$$\sigma^2_{\text{surface}} = -\frac{1}{2\pi^2\Gamma}\int_0^{2\pi}\mathrm{d}\eta_1\left(\frac{\partial}{\partial\eta_1}a(\xi_b,\eta_1)\right)\int_0^{2\pi}\mathrm{d}\eta_2\left(\frac{\partial}{\partial\eta_2}a(\xi_b,\eta_2)\right)\log\left|\sin\frac{\eta_1-\eta_2}{2}\right|$$
$$= \frac{2}{\Gamma}\sum_{n=1}^{\infty}n|a_n|^2 \qquad a(\xi_b,\eta)=\sum_{n=-\infty}^{\infty}a_n\mathrm{e}^{\mathrm{i}n\eta} \tag{13}$$

(note the similarity between (13) and (10)).

There is a simple linear statistic for which the exact distribution can be calculated, thus allowing the above predictions to be tested. This statistic is the linear function $a(x,y) = c_{10}x + c_{01}y$. Substituting in the analogue of (2) for the p.d.f. (11) raised to the power $\Gamma/2$, the resulting dependence on k can be simply calculated by completing the square and changing variables in the integrand (see [5] for an analogous calculation in the case of Hermitian random matrices). We find

$$\tilde{P}(k) = \mathrm{e}^{-k^2(c_{10}^2(1+\tau)+c_{01}^2(1-\tau))/(2\Gamma)} \tag{14}$$

independent of N. Thus $\sigma^2 = \frac{1}{\Gamma}(c_{10}^2(1+\tau)+c_{01}^2(1-\tau))$. Substituting the linear function $a(x,y)$ in (8) and (13) and adding the result verifies that the general formulae reproduce the exact value.

I would like to thank A Soshnikov for detailed discussions, and M Kiessling for comments on the manuscript. This work was supported by the Australian Research Council.

References

[1] Beenakker C W J 1997 *Rev. Mod. Phys.* **69** 731
[2] Brézin E and Zee A 1993 *Nucl. Phys.* B **402** 613
Beenakker C W J 1993 *Phys. Rev.* B **47** 15 763
Beenakker C W J 1994 *Nucl. Phys.* B **422** 515
Jancovici B and Forrester P J 1994 *Phys. Rev.* B **50** 14 599–600
Forrester P J 1995 *Nucl. Phys.* B **435** 421–9
[3] Dyson F J and Mehta M L 1963 *J. Math. Phys.* **4** 701
[4] Johansson K 1988 *Bull. Sci. Math.* 2e *Série* **112** 257
Johansson K 1998 *Duke Math. J.* **91** 151–204
Costin O and Lebowitz J L 1995 *Phys. Rev. Lett.* **75** 69
Basor E L *Commun. Math. Phys.* to appear
Soshnikov A *Ann. Math.* to appear
[5] Baker T H and Forrester P J 1997 *J. Stat. Phys.* **88** 1371–86
[6] Spohn H 1987 *Hydrodynamic Behaviour and Interacting Particle Systems* ed G Papanicolau (Berlin: Springer)
Spohn H 1998 *Preprint* mp_arc/98-198
[7] Ginibre J 1965 *J. Math. Phys.* **6** 440

[8] Fyodorov Y V, Khoruzhenko B A and Sommers H-J 1997 *Phys. Rev. Lett.* **79** 557
[9] Kiessling M and Spohn H 1998 *Preprint* math-phys/9804006
[10] Efetov K B 1997 *Phys. Rev.* B **56** 9630
Janik R A, Novak M A, Papp G and Zahed I 1997 *Preprint* hep-th/9710103
[11] Stephanov M A 1996 *Phys. Rev. Lett.* **76** 4472
Janik R A, Novak M A, Papp G and Zahed I 1996 *Phys. Rev. Lett.* **77** 4876
Halasz M A, Jackson A D and Verbaarschot J J M 1997 *Phys. Lett.* B **395** 293
[12] Alastuey A and Jancovici B 1981 *J. Physique* **42** 1–12
[13] Politzer H D 1989 *Phys. Rev.* B **40** 11 917
Forrester P J *Phys. Rep.* **301** 235–70
[14] Jancovici B 1995 *J. Stat. Phys.* **80** 445
[15] Jancovici B 1982 *J. Stat. Phys.* **29** 263
[16] Choquard Ph, Piller B and Rentsch R 1986 *J. Stat. Phys.* **43** 1975
Choquard Ph, Piller B and Rentsch R 1987 *J. Stat. Phys.* **46** 599–633
Choquard Ph, Piller B, Rentsch R and Vieillefosse P 1989 *J. Stat. Phys.* **55** 1185–262
[17] Di Francesco F, Gaudin M, Itzykson C and Lesage F 1994 *Int. J. Mod. Phys.* A **9** 4257
[18] Forrester P J and Jancovici B 1996 *Int. J. Mod. Phys.* A **11** 941